

ON THE RATE OF CONVERGENCE FOR $(\log_b n)$

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ABSTRACT. In this paper, we study rate of convergence for the distribution of sequence of logarithms $(\log_b n)$ for integer base $b \geq 2$. It is well-known that the slowly growing sequence $(\log_b n)$ is not uniformly distributed modulo one. Its distributions converge to a loop of translated exponential distributions with constant $\log b$ in the space of probabilities on the circle. We give an upper bound for the rate of convergence under Kantorovich distance $d_{\mathbb{T}}$ on the circle. We also give a sharp rate of convergence under Kantorovich distance $d_{\mathbb{R}}$ on the line \mathbb{R} . It turns out that the convergence under $d_{\mathbb{T}}$ is much faster than that under $d_{\mathbb{R}}$.

1. INTRODUCTION

Given a sequence of real numbers $(x_n)_{n=1}^{\infty}$, associate with it a sequence (ν_N) of empirical probability measures

$$\nu_N := \nu_N(x_n) := \frac{1}{N} \sum_{n=1}^N \delta_{\langle x_n \rangle},$$

where δ_a stands for the Dirac measure at $a \in \mathbb{R}$, $\langle a \rangle$ the fractional part of a , and \mathbb{R} the set of real numbers. Note that ν_N is a probability measure on the circle \mathbb{T} , where $\mathbb{T} \equiv [0, 1[$ stands for the unit circle via the canonical transformation $t \mapsto e^{2\pi it}$, and throughout this paper, for $-\infty < a < b \leq \infty$, $[a, b[:= \{x \in \mathbb{R} : a \leq x < b\}$ stands for the half open half closed interval. The intervals $[a, b]$, $]a, b]$, $]a, b[$ are defined analogously. A sequence (x_n) in \mathbb{R} is *uniformly distributed modulo 1* (u.d. mod 1) [5] if

$$\lim_{N \rightarrow \infty} \nu_N([a, b[) = b - a, \quad 0 \leq a < b \leq 1.$$

There are various studies on uniformly distributed sequences modulo 1, say, discrepancies [5]. There are also plenty of sequences not uniformly distributed [11]. For these sequences, *distribution functions* of sequences are a hot topic [5, 10]. The sequence (x_n) is said to have the *distribution function* g if there exists a subsequence $(N_k)_{k=1}^{\infty} \subset \mathbb{N}$ such that $\lim_{k \rightarrow \infty} N_k = \infty$ and

$$\lim_{k \rightarrow \infty} \nu_{N_k}([0, x]) = g(x), \quad \text{for } x \in [0, 1].$$

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An *omega limit set* $\Omega(x_n)$ of a sequence (x_n) is defined by the following set:

$$\Omega(x_n) = \{\mu : \nu_{N_k}(x_n) \rightarrow \mu \text{ weakly for some subsequence } N_k \text{ of } \mathbb{N}\}.$$

One celebrated result on omega limit set of a sequence (or equivalently, distribution functions of a sequence) is due to Winkler [11], stating that the set of (all) distribution functions of a sequence is nonempty, closed and connected, and conversely, any set with these properties can be obtained as the set of distribution functions of some sequence.

One class of sequence not uniformly distributed is the *slowly growing* sequence (x_n) in the sense that [4]

$$(1.1) \quad \lim_{n \rightarrow \infty} n(x_{n+1} - x_n) = \alpha \in [0, \infty[.$$

In [4], it is shown that slowly growing sequences are not uniformly distributed and have a loop of translated exponential distribution functions (for a precise definition of translated exponential distribution functions, we refer the reader to the next section) as the set of distribution functions. Two typical examples of slowly growing sequences are $(\log n)_{n=1}^{\infty}$ and $(\log p_n)_{n=1}^{\infty}$ [10], where $\log(\cdot)$ is the natural logarithm function and (p_n) the sequence of prime numbers.

In [12], distribution functions of $(\log p_n)$ are characterized. In [9], it is shown that (p_n/n) has the same set of distribution functions as $(\log n)$ does. In [3], logarithmically weighted distribution functions of the iterated logarithm $(\log^{(i)} n)$ is studied. In [6], the set of distribution functions of $(f(p_n))$ is characterized for a special class of monotone functions f .

However, there seems no paper addressing how fast a sequence (not uniformly distributed) converges to its set of distribution functions in the literature. One possible way to quantify this convergence is via Kantorovich distance [2], which plays an equivalent role as discrepancies for uniformly distributed sequences. In a recent study [1], an upper bound for the rate of convergence for a class of slowly growing sequences satisfying (1.1) is presented in term of the Kantorovich distance on the circle. However, as it will be seen in Theorem 3.1 later in Section 3, this upper bound, if applied to $(\log_b n)$, is not sharp for any positive integer base $b \geq 2$.

In this paper, we further study this simple slowly growing sequence. We give an improved upper bound for this rate of convergence by directly calculating the Kantorovich distance on the circle. We also obtain the sharp rate of convergence under Kantorovich distance on the line, and show that this convergence is slower than that under Kantorovich distance on the circle.

2. PRELIMINARIES AND NOTATIONS

Let (Y, ρ_Y) be a separable and complete metric space and $\mathcal{P}(Y)$ the family of all Borel probability measures on Y , endowed with the weak topology. Define the *Kantorovich distance* on Y :

$$(2.1) \quad d_Y(\mu, \nu) = \inf \int_{Y \times Y} \rho_Y(x, y) \gamma(dx, dy),$$

for $\mu, \nu \in \mathcal{P}(Y)$, where the infimum is taken over all Borel measurable probability measures on $Y \times Y$ with marginals μ and ν . It is well-known that $d_Y(\cdot, \cdot)$ implies the weak topology on $\mathcal{P}(Y)$. Note that each of $d_{\mathbb{R}}(\nu_N, \nu) \rightarrow 0$ and $d_{\mathbb{T}}(\nu_N, \nu) \rightarrow 0$ implies that $\nu_N \rightarrow \nu$ weakly. For $\mu \in \mathcal{P}(\mathbb{R})$, denote by $\langle \mu \rangle \in \mathcal{P}(\mathbb{T})$ the *induced (or push forward)* probability measure under the mapping $t \mapsto \langle t \rangle$ (recall that $\langle t \rangle$ is the fractional part of $t \in \mathbb{R}$). Let \mathbb{N} be the set of positive integers. For $m, n \in \mathbb{N}$, $m \mid n$ means that n is divisible by m and $m \nmid n$ otherwise. Denote by $\lfloor \cdot \rfloor$ the floor function (i.e., $\lfloor x \rfloor$ is the largest integer not exceeding x), $\lceil \cdot \rceil$ the ceiling function (i.e., $\lceil x \rceil$ is the smallest integer greater than or equal to x), and $\lambda_{\mathbb{T}}$ the Lebesgue measure on \mathbb{T} . Here and hereafter, for convenience, denote by X the compact subspace $[0, 1]$ of \mathbb{R} and let $\rho_X(x, y) = |x - y|$, $\rho_{\mathbb{T}}(x, y) = \min\{|x - y|, 1 - |x - y|\}$, respectively.

For $\mu \in \mathcal{P}(\mathbb{T})$, define

$$F_{\mu}(t) = \mu([0, t]), \quad t \in [0, 1[$$

and

$$F_{\mu}(t-) = \mu([0, t[), \quad t \in [0, 1[.$$

For $s \in \mathbb{T}$, define two probability measures μ_s^D, μ_s^I on X associated with μ by their distribution functions $F_{\mu_s^I}$ and $F_{\mu_s^D}$:

$$F_{\mu_s^I}(t) = \begin{cases} F_{\mu}(t + s) - F_{\mu}(s-), & t \in [0, 1 - s[, \\ 1 + F_{\mu}(t + s - 1) - F_{\mu}(s-), & t \in [1 - s, 1[, \\ 1, & t = 1, \end{cases}$$

and

$$F_{\mu_s^D}(t) = \begin{cases} F_{\mu}(t + s) - F_{\mu}(s), & t \in [0, 1 - s[, \\ 1 + F_{\mu}(t + s - 1) - F_{\mu}(s), & t \in [1 - s, 1]. \end{cases}$$

We identify μ_1^D and μ_1^I with μ_0^D and μ_0^I , respectively.

The following lemma establishes the connection between $d_{\mathbb{T}}$ and d_X , which will be used in the proof of the main result in Section 2.

Lemma 2.1. [13]

$$(2.2) \quad d_{\mathbb{T}}(\mu, \nu) = \min_{s \in \mathbb{T}} \{d_X(\mu_s^D, \nu_s^D), d_X(\mu_s^I, \nu_s^I)\},$$

$$(2.3) \quad d_X(\mu_s^D, \nu_s^D) = \int_0^1 |[F_\mu(t) - F_\nu(t)] - [F_\mu(s) - F_\nu(s)]| dt.$$

For an arbitrary positive integer $b \geq 2$, let $E_b \in \mathcal{P}(X)$ be the exponential distribution of parameter $\log b$ with distribution function

$$F_{E_b}(t) = \frac{b^t - 1}{b - 1}.$$

For convenience, we also use E_b to denote $\langle E_b \rangle$, i.e., we distinguish E_b from $\langle E_b \rangle$ only from the context.

Lemma 2.2. *For $b \geq 2$,*

$$\begin{aligned} F_{E_b \circ R_y^{-1}}(t) &= \begin{cases} \frac{b^{t+y} - b^y}{b - 1}, & t \in [0, 1 - y[\\ 1 + \frac{b^{t+y-1} - b^y}{b - 1}, & t \in [1 - y, 1[\end{cases} \\ &= \langle \frac{b^{<t+y>} - b^y}{b - 1} \rangle, \quad t \in [0, 1], \end{aligned}$$

where R_y is the rotation of \mathbb{T} by $y \in X$. In particular,

$$F_{E_b \circ R_{n - \log_b N}^{-1}}(t) = \begin{cases} \frac{b^n(b^t - 1)}{N(b - 1)}, & t \in [0, \log_b N - n + 1[, \\ 1 + \frac{b^n(b^{t-1} - 1)}{N(b - 1)}, & t \in [\log_b N - n + 1, 1[. \end{cases}$$

Here and hereafter, $\nu_N := \nu_N(\log_b n)$. Let $b^{n-1} \leq N \leq b^n - 1$. Note that $R_{<-\log_b N>} = R_{n - \log_b N}$. Straightforward calculations lead to an explicit formula for F_{ν_N} .

Lemma 2.3. *Let $b^{n-1} \leq N \leq b^n - 1$. Then*

(2.4)

$$F_{\nu_N}(t) = \begin{cases} \frac{n + \sum_{j=0}^{n-1} (\lfloor ib^{-j} \rfloor - b^{n-1-j})}{N}, & t \in [\log_b i - n + 1, \log_b(i + 1) - n + 1[, \\ & i = b^{n-1}, b^{n-1} + 1, \dots, N - 1, \\ \frac{n + \sum_{j=0}^{n-1} (\lfloor Nb^{-j} \rfloor - b^{n-1-j})}{N} = 1 + \frac{n + \sum_{j=0}^{n-1} (\lfloor \lfloor Nb^{-1} \rfloor b^{-j} \rfloor - b^{n-1-j})}{N}, & t \in [\log_b N - n + 1, \log_b(\lfloor Nb^{-1} \rfloor + 1) - n + 2[, \\ 1 + \frac{n + \sum_{j=0}^{n-1} (\lfloor ib^{-j} \rfloor - b^{n-1-j})}{N}, & t \in [\log_b i - n + 2, \log_b(i + 1) - n + 2[, \\ & i = \lfloor Nb^{-1} \rfloor + 1, \lfloor Nb^{-1} \rfloor + 2, \dots, b^n - 1. \end{cases}$$

3. AN UPPER BOUND FOR THE RATE OF CONVERGENCE FOR v_N UNDER $d_{\mathbb{T}}$

In [4], for a sequence (x_n) satisfying (1.1) with $\alpha > 0$,

$$\Omega(x_n) = \{E_{\exp(\alpha^{-1})} \circ R_y^{-1} : y \in \mathbb{T}\}.$$

However, the rate of convergence is not investigated in [4]. In a recent work [1], such rate of convergence is studied and an upper bound is presented.

The following is an application of the result in [1] to the sequence $(\log_b n)$.

Theorem 3.1. [1] *For an arbitrary integer $b \geq 2$,*

$$(3.1) \quad \limsup_{N \rightarrow \infty} \frac{N}{\log N} d_{\mathbb{T}}(v_N, E_b \circ R_{<-\log_b N>}^{-1}) < \infty.$$

The following theorem gives an improved upper bound for $d_{\mathbb{T}}(v_N, E_b \circ R_{<-\log_b N>}^{-1})$, compared with Theorem 3.1.

Theorem 3.2.

$$(3.2) \quad \limsup_{N \rightarrow \infty} \frac{N}{\sqrt{\log N}} d_{\mathbb{T}}(v_N, E_b \circ R_{<-\log_b N>}^{-1}) < \infty.$$

Proof. Without loss of generality (WLOG), we assume $b = 10$ and thus $10^{n-1} \leq N < 10^n - 1$.

We first compute $d_X((v_N)_s^D, E_{10} \circ R_{<-\log_{10} N+s>}^{-1})$ for some particular $s \in \mathbb{T}$ by the formula (2.3), and then we estimate it. Finally, we apply (2.2) in Lemma 2.1 to get an upper bound for $d_{\mathbb{T}}(v_N, E_{10} \circ R_{<-\log_{10} N>}^{-1})$.

By the representation of F_{v_N} in Lemma 2.3, for $s \in [0, \log_{10} N - n + 1[$, there exists a unique $i_0 \in [10^{n-1}, N - 1] \cap \mathbb{N}$ such that

$$s \in [\log_{10} i_0 - n + 1, \log_{10}(i_0 + 1) - n + 1[.$$

Note that $i_0 = \lfloor 10^{s+n-1} \rfloor$. Similarly,

$$(3.3) \quad t \in]\log_{10} i - n + 1, \log_{10}(i + 1) - n + 1[\Leftrightarrow i = \lfloor 10^{t+n-1} \rfloor.$$

$$\begin{aligned} & d_X((v_N)_s^D, (E_{10} \circ R_{<-\log_{10} N>}^{-1})_s^D) \\ &= \sum_{i=10^{n-1}}^{N-1} \int_{\log_{10} i - n + 1}^{\log_{10}(i+1) - n + 1} \left| \frac{\sum_{j=0}^{n-1} (\lfloor i 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor)}{N} - \frac{10^n(10^t - 10^s)}{9N} \right| dt \\ &+ \int_{\log_{10}(N/10) - n + 2}^{\log_{10}(\lfloor N/10 \rfloor + 1) - n + 2} \left| \frac{\sum_{j=0}^{n-1} (\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor)}{N} - \frac{10^n(10^{t-1} - 10^s)}{9N} \right| dt \\ &+ \sum_{i=\lfloor N/10 \rfloor + 1}^{10^{n-1}-1} \int_{\log_{10} i - n + 2}^{\log_{10}(i+1) - n + 2} \left| 1 + \frac{\sum_{j=0}^{n-1} (\lfloor i 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor)}{N} \right. \\ &\quad \left. - \left[1 + \frac{10^n(10^{t-1} - 1)}{9N} - \frac{10^n(10^s - 1)}{9N} \right] \right| dt \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=10^{n-1}}^{N-1} \int_{\log_{10} i - n + 1}^{\log_{10}(i+1) - n + 1} \left| \frac{\sum_{j=0}^{n-1} (\lfloor i 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor)}{N} - \frac{10^n(10^t - 10^s)}{9N} \right| dt \\
&\quad + \int_{\log_{10}(N/10) - n + 1}^{\log_{10}(\lfloor N/10 \rfloor + 1) - n + 1} \left| \frac{\sum_{j=0}^{n-1} (\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor)}{N} - \frac{10^n(10^t - 10^s)}{9N} \right| dt \\
&\quad + \sum_{i=\lfloor N/10 \rfloor + 1}^{10^{n-1}-1} \int_{\log_{10} i - n + 1}^{\log_{10}(i+1) - n + 1} \left| \frac{\sum_{j=0}^{n-1} (\lfloor i 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor)}{N} - \frac{10^n(10^t - 10^s)}{9N} \right| dt \\
&= \sum_{i=\lfloor N/10 \rfloor + 1}^{N-1} \int_{\log_{10} i - n + 1}^{\log_{10}(i+1) - n + 1} \left| \frac{\sum_{j=0}^{n-1} (\lfloor i 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor)}{N} - \frac{10^n(10^t - 10^s)}{9N} \right| dt \\
&\quad + \int_{\log_{10}(N/10) - n + 1}^{\log_{10}(\lfloor N/10 \rfloor + 1) - n + 1} \left| \frac{\sum_{j=0}^{n-1} (\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor)}{N} - \frac{10^n(10^t - 10^s)}{9N} \right| dt.
\end{aligned}$$

Since for all $x \geq 0$,

$$\log(1+x) \leq x, \quad x \geq 0,$$

we have

$$\begin{aligned}
0 &\leq \int_{\log_{10}(N/10) - n + 1}^{\log_{10}(\lfloor N/10 \rfloor + 1) - n + 1} \left| \frac{\sum_{j=0}^{n-1} (\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor)}{N} - \frac{10^n(10^t - 10^s)}{9N} \right| dt \\
&\leq 2 \log_{10} \left(\frac{\lfloor N/10 \rfloor + 1}{N/10} \right) \leq \frac{20}{N^{-1} \log 10}.
\end{aligned}$$

By (3.3),

$$\begin{aligned}
&\frac{\sum_{j=0}^{n-1} (\lfloor i 10^{-j} \rfloor - \lfloor i_0 10^{-j} \rfloor)}{N} - \frac{10^n(10^t - 10^s)}{9N} \\
&= \frac{\sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)]}{N} \\
&\quad - \frac{\sum_{j=0}^{n-1} (i_0 - i) 10^{-j}}{N} - \frac{10^n(10^t - 10^s)}{9N} \\
&= \frac{\sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)]}{N} \\
&\quad + \frac{(1 - 10^{-n})(i - i_0)}{(1 - 10^{-1})N} - \frac{10^n(10^t - 10^s)}{9N} \\
&= \frac{\sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)]}{N}
\end{aligned}$$

$$+ \frac{10[(\lfloor 10^{t+n-1} \rfloor - 10^{t+n-1}) - (\lfloor 10^{s+n-1} \rfloor - 10^{s+n-1})]}{9N} \\ - \frac{10^{-n}(\lfloor 10^{t+n-1} \rfloor - \lfloor 10^{s+n-1} \rfloor)}{(1 - 10^{-1})N}.$$

Also

$$\left| \frac{10[(\lfloor 10^{t+n-1} \rfloor - 10^{t+n-1}) - (\lfloor 10^{s+n-1} \rfloor - 10^{s+n-1})]}{9N} \right. \\ \left. - \frac{10^{-n}(\lfloor 10^{t+n-1} \rfloor - \lfloor 10^{s+n-1} \rfloor)}{(1 - 10^{-1})N} \right| \\ \leq \frac{10|\lfloor 10^{t+n-1} \rfloor - 10^{t+n-1}|}{9N} + \frac{10|\lfloor 10^{s+n-1} \rfloor - 10^{s+n-1}|}{9N} \\ + \frac{10^{-n+1}(10^{t+n-1} + 10^{s+n-1})}{9N} \\ \leq \frac{10}{9N} + \frac{10}{9N} + \frac{20}{9N} = \frac{40}{9N}.$$

Thus for $s \in [0, \log_{10} N - n + 1[$,

$$(3.4) \quad d_X((v_N)_s^D, (E_{10} \circ R_{n-\log_{10} N}^{-1})_s^D) \\ = \frac{1}{N} \sum_{i=\lfloor N/10 \rfloor + 1}^{N-1} \log_{10}(1 + \frac{1}{i}) \cdot \left| \sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) \right. \\ \left. - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)] \right| + O(N^{-1}).$$

In the following, we further estimate the right hand side of (3.4). Since for all $x \geq 0$,

$$0 \leq x - \log(x + 1) \leq x^2/2,$$

we have

$$\frac{1}{i \log 10} - \frac{1}{2i^2 \log 10} \leq \log_{10}(1 + \frac{1}{i}) \leq \frac{1}{i \log 10}.$$

Note that

$$\sum_{i=\lfloor N/10 \rfloor + 1}^{N-1} \frac{1}{2i^2 \log 10} \left| \sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)] \right| \\ \leq \frac{1}{2 \log 10} \sum_{i=\lfloor N/10 \rfloor + 1}^{N-1} \frac{1}{i^2} \cdot 2n \leq \frac{n}{\log 10} \sum_{i=\lfloor N/10 \rfloor + 1}^{N-1} \frac{1}{(i-1)i} \leq \frac{n}{\log 10} \frac{1}{\lfloor N/10 \rfloor},$$

i.e.,

$$\sum_{i=\lfloor N/10 \rfloor + 1}^{N-1} \frac{1}{2i^2 \log 10} \left| \sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)] \right| \\ = O(nN^{-1}).$$

By triangle inequality,

$$d_X((v_N)_s^D, (E_{10} \circ R_{n-\log_{10} N}^{-1})_s^D) \\ = \frac{1}{N \log 10} \sum_{i=\lfloor N/10 \rfloor + 1}^{N-1} \frac{1}{i} \left| \sum_{j=0}^{n-1} [i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor] - (i 10^{-j} - \lfloor i 10^{-j} \rfloor) \right| \\ + O(N^{-1}).$$

For $i \geq \lfloor N/10 \rfloor$,

$$\frac{1}{N \log 10} \frac{1}{i} \left| \sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)] \right| \leq \frac{2n}{N \lfloor N/10 \rfloor \log 10},$$

we have

$$d_X((v_N)_s^D, (E_{10} \circ R_{n-\log_{10} N}^{-1})_s^D) \\ (3.5) \quad = \frac{1}{N \log 10} \sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i} \left| \sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)] \right| \\ + O(N^{-1}).$$

Similarly, for $s \in [\log_{10} N - n + 1, 1[$, there exists a unique $i_0 \in [\lfloor N/10 \rfloor + 1, 10^{n-1} - 1] \cap \mathbb{N}$ such that

$$\log_{10} i_0 - n + 2 \leq s < \log_{10}(i_0 + 1) - n + 2$$

with $i_0 = \lfloor 10^{s+n-2} \rfloor$. For this case, we still have the same asymptotic expansion (3.5). The rest is to estimate $\sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i} \left| \sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)] \right|$.

By Cauchy-Schwarz inequality, we have

$$(3.6) \quad \left(\sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i} \left| \sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)] \right| \right)^2 \\ \leq \sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i^2} \sum_{i=\lfloor N/10 \rfloor}^N \left(\sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)] \right)^2$$

Note that

$$(3.7) \quad \sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i^2} \leq \sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{(i-1)i} = 9N^{-1} + O(N^{-2}).$$

Similarly, we have that

$$(3.8) \quad \sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i^2} \geq 9N^{-1} + O(N^{-2}).$$

Inequalities (3.7) and (3.8) together imply that

$$(3.9) \quad \sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i^2} = 9N^{-1} + O(N^{-2}).$$

Now we study the asymptotics of

$$\sum_{i=0}^N \left(\sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)] \right)^2.$$

Notice that

$$\begin{aligned} & \sum_{i=0}^N \left(\sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)] \right)^2 \\ &= \sum_{i=0}^N \sum_{j=0}^{n-1} [(i 10^{-j} - \lfloor i 10^{-j} \rfloor)]^2 - 2 \sum_{i=0}^N \sum_{j=0}^{n-1} (i 10^{-j} - \lfloor i 10^{-j} \rfloor) \\ & \quad \cdot \sum_{j=0}^{n-1} (i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) + \sum_{i=0}^N \left[\sum_{j=0}^{n-1} (i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) \right]^2 \\ &= \sum_{i=0}^N \left[\sum_{j=0}^{n-1} (i 10^{-j} - \lfloor i 10^{-j} \rfloor) \right]^2 - 2 \sum_{i=0}^N \left[\sum_{j=0}^{n-1} (i 10^{-j} - \lfloor i 10^{-j} \rfloor) \right] \\ & \quad \cdot \left[\sum_{j=0}^{n-1} (i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) \right] + (N+1) \left[\sum_{j=0}^{n-1} (i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) \right]^2. \end{aligned} \quad (3.10)$$

In order to get the desired estimate, we need to obtain asymptotic expansions for the three following terms separately as $N \rightarrow \infty$:

- (1) $\sum_{i=0}^N \sum_{j=0}^{n-1} (i 10^{-j} - \lfloor i 10^{-j} \rfloor).$
- (2) $\sum_{i=0}^N \left[\sum_{j=0}^{n-1} (i 10^{-j} - \lfloor i 10^{-j} \rfloor) \right]^2.$
- (3) $\sum_{j=0}^{n-1} (i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor).$

$$(1) \sum_{i=0}^N \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor).$$

For $j = 0, \dots, n-1$, for any $0 \leq i \leq N$, there exists a unique pair of nonnegative integers k, l with $l \leq 10^j - 1$ such that $i = k10^j + l$. Note that

$$\begin{aligned} \{i\}_{i=0}^N &= \{k10^j + l \mid 0 \leq k \leq \lfloor N10^{-j} \rfloor - 1, \\ &\quad 0 \leq l \leq 10^j - 1 \\ &\quad \cup \{\lfloor N10^{-j} \rfloor 10^j + l \mid 0 \leq l \leq N - \lfloor N10^{-j} \rfloor 10^j\} \end{aligned}$$

and

$$i10^{-j} - \lfloor i10^{-j} \rfloor = l10^{-j}.$$

Let $\sum_{j=0}^{n-1} a_j 10^j$ be the decimal expansion for N . Thus

$$\lfloor N10^{-j} \rfloor = N10^{-j} - \sum_{r=0}^{j-1} a_r 10^{r-j}, \text{ for } j = 1, \dots, n-1.$$

By

$$\sum_{j=0}^{n-1} (1 - 10^{-j}) = n - \frac{10(1 - 10^{-n})}{9}$$

and

$$\sum_{j=1}^{n-1} 10^{-j} \sum_{r=0}^{j-1} a_r 10^r \leq \sum_{j=1}^{n-1} 1 = n - 1 = O(n),$$

we have

(3.11)

$$\begin{aligned} &\sum_{i=0}^N \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) = \sum_{j=0}^{n-1} \sum_{i=0}^N (i10^{-j} - \lfloor i10^{-j} \rfloor) \\ &= \sum_{j=0}^{n-1} \left[\sum_{k=0}^{\lfloor N10^{-j} \rfloor - 1} \sum_{l=0}^{10^j - 1} l10^{-j} + \sum_{l=0}^{N - \lfloor N10^{-j} \rfloor 10^j} l10^{-j} \right] \\ &= \frac{1}{2} \sum_{j=0}^{n-1} [(10^j - 1)\lfloor N10^{-j} \rfloor + (N10^{-j} - \lfloor N10^{-j} \rfloor)(1 + N - \lfloor N10^{-j} \rfloor 10^j)] \\ &= \frac{1}{2} \sum_{j=0}^{n-1} N(1 - 10^{-j}) - \frac{1}{2} \sum_{j=1}^{n-1} \sum_{r=0}^{j-1} a_r 10^r + \sum_{j=1}^{n-1} 10^{-j} \sum_{r=0}^{j-1} a_r 10^r \\ &\quad + \frac{1}{2} \sum_{j=1}^{n-1} 10^{-j} \left(\sum_{r=0}^{j-1} a_r 10^r \right)^2 \end{aligned}$$

$$= \frac{1}{2}(n - \frac{10}{9})N - \frac{1}{2} \sum_{j=1}^{n-1} \sum_{r=0}^{j-1} a_r 10^r + \frac{1}{2} \sum_{j=1}^{n-1} 10^{-j} \left(\sum_{r=0}^{j-1} a_r 10^r \right)^2 + O(n).$$

(2) $\sum_{i=0}^N [\sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor)]^2$. We first expand it as follows.

$$\begin{aligned} & \sum_{i=0}^N \left[\sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \right]^2 \\ &= \sum_{i=0}^N \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \sum_{r=0}^{n-1} (i10^{-r} - \lfloor i10^{-r} \rfloor) \\ &= 2 \sum_{i=0}^N \sum_{j=1}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \sum_{r=0}^{j-1} (i10^{-r} - \lfloor i10^{-r} \rfloor) \\ & \quad + \sum_{i=0}^N \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor)^2. \end{aligned}$$

It now reduces to getting the asymptotic expansion for

$$\sum_{i=0}^N \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor)^2$$

and

$$\sum_{i=0}^N \sum_{j=1}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \sum_{r=0}^{j-1} (i10^{-r} - \lfloor i10^{-r} \rfloor),$$

respectively.

For $\sum_{i=0}^N \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor)^2$, notice that

$$\begin{aligned} & \sum_{i=0}^N \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor)^2 = \sum_{j=0}^{n-1} \sum_{i=0}^N (i10^{-j} - \lfloor i10^{-j} \rfloor)^2 \\ &= \sum_{j=0}^{n-1} \left[\sum_{i=0}^{\lfloor N10^{-j} \rfloor 10^j - 1} (i10^{-j} - \lfloor i10^{-j} \rfloor)^2 + \sum_{i=\lfloor N10^{-j} \rfloor 10^j}^N (i10^{-j} - \lfloor i10^{-j} \rfloor)^2 \right] \\ &= \sum_{j=0}^{n-1} \left[\sum_{k=0}^{\lfloor N10^{-j} \rfloor - 1} \sum_{l=0}^{10^j - 1} (l10^{-j})^2 + \sum_{l=0}^{N - \lfloor N10^{-j} \rfloor 10^j} (l10^{-j})^2 \right] \\ &= \frac{1}{6} \sum_{j=0}^{n-1} \left\{ (10^j - 1)(2 - 10^{-j}) \lfloor N10^{-j} \rfloor + (10^{-j} + N10^{-j} \right. \\ & \quad \left. - \lfloor N10^{-j} \rfloor)(N10^{-j} - \lfloor N10^{-j} \rfloor)[2(N - \lfloor N10^{-j} \rfloor 10^j) + 1] \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{6} \sum_{j=0}^{n-1} (2 \cdot 10^j - 3 + 10^{-j}) \lfloor N10^{-j} \rfloor - \frac{1}{6} \sum_{j=1}^{n-1} (10^{-j} + N10^{-j} - \lfloor N10^{-j} \rfloor) \\
&\quad \cdot (N10^{-j} - \lfloor N10^{-j} \rfloor) [2 \cdot 10^j (N10^{-j} - \lfloor N10^{-j} \rfloor) + 1].
\end{aligned}$$

It is straightforward to obtain

$$\begin{aligned}
0 &\leq \sum_{j=1}^{n-1} (10^{-j} + N10^{-j} - \lfloor N10^{-j} \rfloor) (N10^{-j} - \lfloor N10^{-j} \rfloor) \\
&\quad \cdot [2(N - \lfloor N10^{-j} \rfloor 10^j) + 1] \leq \sum_{j=1}^{n-1} (10^{-j} + 1) \cdot 1 \cdot [2 \cdot 10^j + 1] \\
&\leq 4n + \frac{2(10^n - 1)}{9},
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\sum_{j=1}^{n-1} (10^{-j} + N10^{-j} - \lfloor N10^{-j} \rfloor) (N10^{-j} - \lfloor N10^{-j} \rfloor) \\
&\quad \cdot [2(N - \lfloor N10^{-j} \rfloor 10^j) + 1] = O(N).
\end{aligned}$$

Also

$$0 \leq \sum_{j=1}^{n-1} (3 - 10^{-j}) \lfloor N10^{-j} \rfloor \leq \frac{N}{3},$$

i.e.,

$$\sum_{j=1}^{n-1} (3 - 10^{-j}) \lfloor N10^{-j} \rfloor = O(N).$$

Since

$$\sum_{j=0}^{n-1} 10^j (N10^{-j} - 1) = nN + O(N),$$

we have $\sum_{j=0}^{n-1} 10^j \lfloor N10^{-j} \rfloor = nN + O(N)$. This implies that

$$\sum_{i=0}^N \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor)^2 = \frac{nN}{3} + O(N).$$

Now we obtain the asymptotic expansion for the mixed term

$$\sum_{i=0}^N \sum_{j=1}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \sum_{r=0}^{j-1} (i10^{-r} - \lfloor i10^{-r} \rfloor).$$

For $j = 1, \dots, n-1$, $r = 0, \dots, j-1$, for any $0 \leq i \leq N$, there exists a unique triple of nonnegative integers k, p, l with $p \leq 10^{j-r}-1$, $l \leq 10^r-1$ such that $i = k10^j + p10^r + l$. Then

$$\begin{aligned} \{i\}_{i=0}^N &= \{k10^j + p10^r + l\} \quad \begin{array}{l} 0 \leq k \leq \lfloor N10^{-j} \rfloor - 1 \\ 0 \leq p \leq 10^{j-r} - 1 \\ 0 \leq l \leq 10^r - 1 \end{array} \\ &\cup \{\lfloor N10^{-j} \rfloor 10^j + p10^r + l\} \quad \begin{array}{l} 0 \leq p \leq \lfloor K_j 10^{-r} \rfloor - 1 \\ 0 \leq l \leq 10^r - 1 \end{array} \\ &\cup \{\lfloor N10^{-j} \rfloor 10^j + \lfloor K_j 10^{-r} \rfloor 10^r + l\} \quad 0 \leq l \leq K_j - \lfloor K_j 10^{-r} \rfloor 10^r, \end{aligned}$$

where $K_j = N - \lfloor N10^{-j} \rfloor 10^j$.

We rewrite $\sum_{i=0}^N \sum_{j=1}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \sum_{r=0}^{j-1} (i10^{-r} - \lfloor i10^{-r} \rfloor)$ as follows:

$$\begin{aligned} &\sum_{i=0}^N \sum_{j=1}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \sum_{r=0}^{j-1} (i10^{-r} - \lfloor i10^{-r} \rfloor) \\ &= \sum_{j=1}^{n-1} \sum_{r=0}^{j-1} \left[\sum_{i=0}^{\lfloor N10^{-j} \rfloor 10^{j-1}} (i10^{-j} - \lfloor i10^{-j} \rfloor)(i10^{-r} - \lfloor i10^{-r} \rfloor) \right. \\ &\quad + \sum_{i=\lfloor N10^{-j} \rfloor 10^j}^{\lfloor N10^{-j} \rfloor 10^j + \lfloor K_j 10^{-r} \rfloor 10^r - 1} (i10^{-j} - \lfloor i10^{-j} \rfloor)(i10^{-r} - \lfloor i10^{-r} \rfloor) \\ &\quad \left. + \sum_{i=\lfloor N10^{-j} \rfloor 10^j + \lfloor K_j 10^{-r} \rfloor 10^r}^N (i10^{-j} - \lfloor i10^{-j} \rfloor)(i10^{-r} - \lfloor i10^{-r} \rfloor) \right] \\ &= \sum_{j=1}^{n-1} \sum_{r=0}^{j-1} \left[\sum_{k=0}^{\lfloor N10^{-j} \rfloor - 1} \sum_{p=0}^{10^{j-r}-1} \sum_{l=0}^{10^r-1} (p10^r + l)10^{-j}l10^{-r} \right. \\ &\quad + \sum_{p=0}^{\lfloor K_j 10^{-r} \rfloor - 1} \sum_{l=0}^{10^r-1} (p10^r + l)10^{-j}l10^{-r} \\ &\quad \left. + \sum_{l=0}^{K_j - \lfloor K_j 10^{-r} \rfloor 10^r} (\lfloor K_j 10^{-r} \rfloor 10^r + l)10^{-j}l10^{-r} \right]. \end{aligned}$$

By tedious but straightforward calculations, we have

$$\sum_{i=0}^N \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \sum_{r=0}^{j-1} (i10^{-r} - \lfloor i10^{-r} \rfloor)$$

$$= \left(\frac{n^2}{8} - \frac{85n}{216} \right) N - \frac{1}{4} \sum_{j=1}^{n-1} j \sum_{l=0}^{j-1} a_l 10^l + \frac{1}{4} \sum_{j=1}^{n-1} j 10^{-j} \left[\sum_{l=0}^{j-1} a_l 10^l \right]^2 + O(N),$$

where we recall that $\sum_{j=0}^{n-1} a_j 10^j$ is the decimal expansion for N . This further implies that

(3.12)

$$\begin{aligned} & \sum_{i=0}^N \left[\sum_{j=0}^{n-1} (i 10^{-j} - \lfloor i 10^{-j} \rfloor) \right]^2 \\ &= \left(\frac{n^2}{4} - \frac{49}{108} n \right) N - \frac{1}{2} \sum_{j=0}^{n-1} j \sum_{l=0}^{j-1} a_l 10^l + \frac{1}{2} \sum_{j=0}^{n-1} j 10^{-j} \left[\sum_{l=0}^{j-1} a_l 10^l \right]^2 + O(N). \end{aligned}$$

Next we choose proper i_0 and obtain asymptotic expansion for $\sum_{j=0}^{n-1} (i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor)$ in (3.10).

(3) $\sum_{j=0}^{n-1} (i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor)$. Define

$$i_0 = \begin{cases} 10^{n-1} - 10^{\lfloor n/2 \rfloor - 1} + 1, & N \in [10^{n-1}, 10^n - 10^{\lfloor n/2 \rfloor}] \cap \mathbb{N}, \\ 10^n - 10^{\lfloor n/2 \rfloor}, & N \in]10^n - 10^{\lfloor n/2 \rfloor}, 10^n - 1] \cap \mathbb{N}. \end{cases}$$

Note that $i_0 \in [\lfloor N/10 \rfloor + 1, N - 1] \cap \mathbb{N}$. Then

(a.1) $N \in [10^{n-1}, 10^n - 10^{\lfloor n/2 \rfloor}] \cap \mathbb{N}$ and $i_0 = 10^{n-1} - 10^{\lfloor n/2 \rfloor - 1}$. It is easy to verify that

$$\sum_{j=0}^{n-1} (i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) = \begin{cases} \frac{n}{2}, & 2 \mid n \\ \frac{n+1}{2}, & 2 \nmid n \end{cases} + O(N^{-1/2}).$$

(a.2) $N \in]10^n - 10^{\lfloor n/2 \rfloor}, 10^n - 1] \cap \mathbb{N}$ and $i_0 = 10^n - 10^{\lfloor n/2 \rfloor}$. Similarly, we have

$$\sum_{j=0}^{n-1} (i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) = \begin{cases} \frac{n}{2} - \frac{10}{9}, & 2 \mid n \\ \frac{n}{2} - \frac{11}{18}, & 2 \nmid n \end{cases} + O(N^{-1/2}).$$

In sum, for $10^{n-1} \leq N \leq 10^n - 1$, we can choose $i_0 \in [\lfloor N/10 \rfloor + 1, N - 1] \cap \mathbb{N}$ such that

$$(3.13) \quad \sum_{j=0}^{n-1} (i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) = \frac{n}{2} + c + O(N^{-1/2})$$

for some constant c .

Thus, by (3.11), (3.12) and (3.13), we have

$$\begin{aligned}
& \sum_{i=0}^N \left[\sum_{j=0}^{n-1} [(i10^{-j} - \lfloor i10^{-j} \rfloor) - (i_010^{-j} - \lfloor i_010^{-j} \rfloor)] \right]^2 \\
&= \left[\frac{11}{108}N + \frac{1}{2} \sum_{j=0}^{n-1} \sum_{l=0}^{j-1} a_l 10^l - \frac{1}{2} \sum_{j=0}^{n-1} 10^{-j} \left(\sum_{l=0}^{j-1} a_l 10^l \right)^2 \right] n \\
&\quad - \frac{1}{2} \sum_{j=0}^{n-1} j \sum_{l=0}^{j-1} a_l 10^l + \frac{1}{2} \sum_{j=0}^{n-1} j 10^{-j} \left(\sum_{l=0}^{j-1} a_l 10^l \right)^2 + O(N) \\
&= \frac{11}{108}nN + \frac{1}{2} \sum_{j=0}^{n-1} (n-j) \left(\sum_{l=0}^{j-1} a_l 10^l \right) \left(1 - \sum_{l=0}^{j-1} a_l 10^{l-j} \right) + O(N).
\end{aligned}$$

It is straightforward to see that

$$\frac{1}{2} \sum_{j=0}^{n-1} (n-j) \left(\sum_{l=0}^{j-1} a_l 10^l \right) \left(1 - \sum_{l=0}^{j-1} a_l 10^{l-j} \right) = O(N).$$

Therefore, we have

$$\begin{aligned}
& \sum_{j=0}^N \left\{ \sum_{j=0}^{n-1} [(i10^{-j} - \lfloor i10^{-j} \rfloor) - (i_010^{-j} - \lfloor i_010^{-j} \rfloor)] \right\}^2 \\
&= \frac{11}{108}nN + O(N).
\end{aligned}$$

Hence

$$\begin{aligned}
& \sum_{j=\lfloor N/10 \rfloor}^N \left\{ \sum_{j=0}^{n-1} [(i10^{-j} - \lfloor i10^{-j} \rfloor) - (i_010^{-j} - \lfloor i_010^{-j} \rfloor)] \right\}^2 \\
&= \frac{11}{108} \frac{9}{10} nN + O(N) = \frac{11 \log N}{120 \log 10} N + O(N).
\end{aligned}$$

By (3.5), (3.6), (3.9) and letting $s = \lfloor \log_{10} i_0 \rfloor$, we have

$$\begin{aligned}
& \limsup_{N \rightarrow \infty} \frac{N}{\sqrt{\log N}} d_X((v_N)_{<\log_{10} i_0>}^D, (E_{10} \circ R_{n-\log_{10} N}^{-1})_{<\log_{10} i_0>}^D) \\
&\leq \frac{1}{\log 10} \sqrt{\frac{33}{40 \log 10}}.
\end{aligned}$$

By (2.2),

$$\limsup_{N \rightarrow \infty} \frac{N}{\sqrt{\log N}} d_T(v_N, E_{10} \circ R_{n-\log_{10} N}^{-1}) \leq \frac{1}{\log 10} \sqrt{\frac{33}{40 \log 10}}.$$

We complete the proof. \square

Remark 3.3. If replacing Cauchy-Schwarz inequality (3.6) by the following Hölder inequality

$$\begin{aligned} & \left| \sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i} \sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)] \right| \\ & \leq \left(\sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i^{4/3}} \right)^{3/4} \left(\sum_{i=\lfloor N/10 \rfloor}^N \left(\sum_{j=0}^{n-1} [(i_0 10^{-j} - \lfloor i_0 10^{-j} \rfloor) - (i 10^{-j} - \lfloor i 10^{-j} \rfloor)] \right)^4 \right)^{1/4}, \end{aligned}$$

we have the same upper bound for the rate of convergence

$$\limsup_{N \rightarrow \infty} \frac{N}{\sqrt{\log N}} d_{\mathbb{T}}(\nu_N, E_b \circ R_{<-\log_b N}^{-1}) \leq c$$

but with a positive constant $c < \frac{1}{\log 10} \sqrt{\frac{33}{40 \log 10}}$. Thus we conjecture

Conjecture 3.4.

$$\begin{aligned} 0 & < \liminf_{N \rightarrow \infty} \frac{N}{\sqrt{\log N}} d_{\mathbb{T}}(\nu_N, E_b \circ R_{<-\log_b N}^{-1}) \\ & \leq \limsup_{N \rightarrow \infty} \frac{N}{\sqrt{\log N}} d_{\mathbb{T}}(\nu_N, E_b \circ R_{<-\log_b N}^{-1}) < \infty. \end{aligned}$$

Remark 3.5. By Zador's theorem on quantization error for probability measures on the circle [13], we know that

$$\liminf_{N \rightarrow \infty} N d_{\mathbb{T}}(\nu_N, E_b \circ R_{<-\log_b N}^{-1}) > 0.$$

This shows that such convergence cannot be faster than N^{-1} .

4. A SHARP RATE OF CONVERGENCE FOR ν_N UNDER d_X

However, the same convergence, if under a different metric, may change. In this section, we characterize the sharp rate of convergence under the Kantorovich distance on the line and show that the convergence under $d_{\mathbb{T}}$ is much faster than that under d_X .

Theorem 4.1. *For an arbitrary integer $b \geq 2$,*

$$\boxed{\lim_{N \rightarrow \infty} \frac{N}{\log N} d_X(\mu_N, E_b \circ R_{<-\log_b N}^{-1}) = \frac{1}{2 \log b}.}$$

Proof. WLOG, assume $b = 10$. Similar to (3.5) in the proof of Theorem 4.1, it is easy to verify that for $10^{n-1} \leq N \leq 10^n - 1$,

$$(4.1) \quad \begin{aligned} & d_X(v_N, E_{10} \circ R_{n-\log_{10} N}^{-1}) \\ &= \frac{1}{N \log 10} \sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i} \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) + O(N^{-1}). \end{aligned}$$

Like the representation for $\sum_{i=0}^N \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor)$ in the proof of Theorem 4.1, we have

$$\begin{aligned} & \sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i} \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \\ &= \sum_{j=0}^{n-1} \left[\sum_{i=\lfloor N/10 \rfloor}^{\lfloor \lfloor N/10 \rfloor 10^{-j} + 1 \rfloor 10^j - 1} \frac{i10^{-j} - \lfloor i10^{-j} \rfloor}{i} + \sum_{i=(\lfloor \lfloor N/10 \rfloor 10^{-j} + 1 \rfloor 10^j)}^{\lfloor N/10 \rfloor 10^j - 1} \frac{i10^{-j} - \lfloor i10^{-j} \rfloor}{i} \right. \\ & \quad \left. + \sum_{i=\lfloor N/10 \rfloor 10^j}^N \frac{i10^{-j} - \lfloor i10^{-j} \rfloor}{i} \right] \\ &= \sum_{j=0}^{n-1} \left[\sum_{l=\lfloor N/10 \rfloor - \lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor 10^j}^{10^j - 1} \frac{l10^{-j}}{\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor 10^j + l} \right. \\ & \quad \left. + \sum_{k=\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor + 1}^{\lfloor N/10 \rfloor 10^j - 1} \sum_{l=0}^{10^j - 1} \frac{l10^{-j}}{k10^j + l} + \sum_{l=0}^{N - \lfloor N/10 \rfloor 10^j} \frac{l10^{-j}}{\lfloor N/10 \rfloor 10^j + l} \right]. \end{aligned}$$

On one hand,

$$(4.2) \quad \begin{aligned} & \sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i} \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \\ & \geq \sum_{j=0}^{n-1} \left[\sum_{l=\lfloor N/10 \rfloor - \lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor 10^j}^{10^j - 1} \frac{l10^{-j}}{\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor 10^j + 10^j} \right. \\ & \quad \left. + \sum_{k=\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor + 1}^{\lfloor N/10 \rfloor 10^j - 1} \sum_{l=0}^{10^j - 1} \frac{l10^{-j}}{k10^j + 10^j} + \sum_{l=0}^{N - \lfloor N/10 \rfloor 10^j} \frac{l10^{-j}}{N} \right] \\ &= \frac{1}{2} \sum_{j=0}^{n-1} \left[\frac{(1 - 10^{-j} + \lfloor N/10 \rfloor 10^{-j} - \lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor)}{\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor + 1} \right] \end{aligned}$$

$$\begin{aligned} & \cdot (1 - \lfloor N/10 \rfloor 10^{-j} + \lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor) + (1 - 10^{-j}) \sum_{k=\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor + 1}^{\lfloor N 10^{-j} \rfloor - 1} \frac{1}{k+1} \\ & + \frac{(N 10^{-j} - \lfloor N 10^{-j} \rfloor)(N 10^{-j} - \lfloor N 10^{-j} \rfloor + 1) 10^j}{N} \Big]. \end{aligned}$$

It is easy to see that

$$\begin{aligned} & \frac{1}{2} \sum_{j=0}^{n-1} \left[\frac{(1 - 10^{-j} + \lfloor N/10 \rfloor 10^{-j} - \lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor)}{\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor + 1} \right. \\ & \cdot (1 - \lfloor N/10 \rfloor 10^{-j} + \lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor) \\ & \left. + \frac{(N 10^{-j} - \lfloor N 10^{-j} \rfloor)(N 10^{-j} - \lfloor N 10^{-j} \rfloor + 1) 10^j}{N} \right] = O(1). \end{aligned}$$

Since for all $x \geq 0$,

$$\log(1+x) \leq x,$$

we have for $N_1, N_2 \in \mathbb{N}$, $N_2 > N_1 > 1$,

$$\begin{aligned} \sum_{k=N_1}^{N_2} \frac{1}{k} & > \sum_{k=N_1}^{N_2} \int_k^{k+1} \frac{1}{x} dx = \log \frac{N_2+1}{N_1}, \\ \sum_{k=N_1}^{N_2} \frac{1}{k} & < \sum_{k=N_1}^{N_2} \int_{k-1}^k \frac{1}{x} dx = \log \frac{N_2}{N_1-1}. \end{aligned}$$

It is straightforward to check that

$$\frac{1}{2} \sum_{j=0}^{n-1} (1 - 10^{-j}) \sum_{k=\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor + 1}^{\lfloor N 10^{-j} \rfloor - 1} \frac{1}{k+1} = \frac{\log 10}{2} n + O(1).$$

Thus by (4.2),

$$\sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i} \sum_{j=0}^{n-1} (i 10^{-j} - \lfloor i 10^{-j} \rfloor) \geq \frac{\log 10}{2} n + O(1).$$

On the other hand,

$$\begin{aligned}
& \sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i} \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \\
& \leq \sum_{j=0}^{n-1} \left[\sum_{l=\lfloor N/10 \rfloor - \lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor 10^j}^{10^j-1} \frac{l10^{-j}}{\lfloor N/10 \rfloor} \right. \\
& \quad \left. + \sum_{k=\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor + 1}^{\lfloor N10^{-j} \rfloor - 1} \sum_{l=0}^{10^j-1} \frac{l10^{-j}}{k10^j} + \sum_{l=0}^{N - \lfloor N10^{-j} \rfloor 10^j} \frac{l10^{-j}}{\lfloor N10^{-j} \rfloor 10^j} \right] \\
& = \frac{1}{2} \sum_{j=0}^{n-1} \left[\frac{(1 - 10^{-j} + \lfloor N/10 \rfloor 10^{-j} - \lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor) 10^j}{\lfloor N/10 \rfloor} \right. \\
& \quad \cdot (1 - \lfloor N/10 \rfloor 10^{-j} + \lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor) + (1 - 10^{-j}) \sum_{k=\lfloor \lfloor N/10 \rfloor 10^{-j} \rfloor + 1}^{\lfloor N10^{-j} \rfloor - 1} \frac{1}{k} \\
& \quad \left. + \frac{(N10^{-j} - \lfloor N10^{-j} \rfloor)(N10^{-j} - \lfloor N10^{-j} \rfloor + 1)}{\lfloor N10^{-j} \rfloor} \right].
\end{aligned}$$

Using similar arguments, we can show

$$\sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i} \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) \leq \frac{\log 10}{2} n + O(1).$$

In sum,

$$\sum_{i=\lfloor N/10 \rfloor}^N \frac{1}{i} \sum_{j=0}^{n-1} (i10^{-j} - \lfloor i10^{-j} \rfloor) = \frac{\log 10}{2} n + O(1) = \frac{1}{2} \log N + O(1).$$

Then the conclusion follows from (4.1). \square

Remark 4.2. From Theorem 3.1 and Theorem 4.1, we see $d_{\mathbb{T}}(\nu_N, E_b \circ R_{<-\log_b N>}^{-1})$ decays much faster than $d_X(\nu_N, E_b \circ R_{<-\log_b N>}^{-1})$ as $N \rightarrow \infty$.

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